METHODOLOGY FOR COMPUTING SHORT TERM ZERO COUPON CURVE AS OF EURIBOR FUTURE RATES

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8 rue du Mail 75002 Paris

http://www.cnofrance.org

Abstract

The valuation of a series of future cash flows (such as those paid by a fixed-rate bond) requires to calculate the present value of one unit of currency at each payment date.

The computation of a present value is commonly based on the preliminary estimation of a zero-coupon rate at each payment date. Cash flows are then discounted at these rates, possibly further adjusted with credit risk components.

Market data is used to compute zero coupon rates at market maturity dates, i.e. on an annual basis.

The zero-coupon yield curve computed by the CNO/FBA is related to the swap and future yield curve.

So as to calculate the zero-coupon yield curve for a specific month, the FBA uses the last working day's closing rate of this given month.

This historical yield curve is provided by spot values for calendar years maturities comprised between 1 and 60 years.

Intermediate values can be obtained by iteration.

The swap rates represent the coupon rates at par associated with the zero-coupon yield curve.

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Key elements of calculation process

The zero-coupon rate structure is based on the 6 Month-Euribor interest rate swap rates. It is supplied as a series of swap rates with maturities ranging from 1 year to 60 years. Such swap rates are exact par rates.

The zero-coupon rates are calculated by bootstrapping procedure. This procedure requires to pre-calculate the missing annual values of the swap rate structure. The missing values (31 years, 32 years, and 33 years, etc.) are calculated using a cubic interpolation method. This polynomial interpolation function of degree 3 has been chosen to calculate the missing swap rates for a given maturity (included between the two closest annual market rates). It is specific for every interval and it preserves at each extremity of the interval the values of the quoted par swap as well as the values of the first two order derivatives.

For short term maturities (from 3 months up to 3 years) there is an alternative data source to the IRS swap curve consisting of 3-Month Euribor Futures quoted on the Liffe and on Eurex. The Euribor Futures market is highly liquid and Futures prices are in relative terms more reliable than those of the short term swaps.

In order to calculate the discount factors for short term maturities, it is necessary to adjust the data twice. These adjustments consist in a convexity and a liquidity adjustment.

CNO publishes monthly a zero coupon curve for every annual maturity from 1 year up to 60 years. For other maturities, one can use an interpolation method. In most cases a linear interpolation of the zero coupon rate over the maturity can be sufficient. If needed a cubic interpolation similar to that used in the paper is recommended. A polynomial interpolation of upper order is not considered to be relevant.

CNO draws the attention that these rates are not calculated in real time and are not intended to be used in market transactions.

Methodology

Notation:

Maturity (in civil year fraction)	:	**u**
Observed maturity (in civil year fraction)	:	u_i
Swap rate (% annual rate)	:	**Sw(u)**
Observed swap rate (% annual rate)	:	$Sw_i = Sw(u_i)$
Swap rate slope	:	**P(u) =dSw(u) /du**
Observed swap rate slope	:	$P_i = P(u_i)$
Calculated zero coupon rate (% annual rate)	:	**Z(u)**
PV of unit	:	**Va(u) = (1+Z(u)/100)^-u**
PV of annual income	:	$R(u) = \Sigma_{k=1,u} Va(k)$

The calculation date is defined as the date corresponding to the swap rates used in the calculation (and not as the actual calculation date).

1. Sources of Data

Swap Rate

The "observed" swaps rates used in the calculation process are the ones published in annualized % with 3 decimals The observed swap rates are commonly quoted for the following maturities: u_i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 35, 40, 45, 50, 60 years.

The unit of time is the rolling calendar year of 365 or 366 days (from one anniversary date to the other as of the collection date of data).

In absence of swap quotes or in case of obvious data errors and if the omission of these numbers does not affect in a considerable way the quality of the results, the CNO will replace these quotes by interpolated values calculated as of the closest known maturities (according to the interpolation method used for the yearly missing maturities: see below).

If the swap rates are not published, the CNO initially planned calculation date, or if a partial publication could affect the quality of the results, the CNO calculation date would be moved to the first preceding day for which data is available.

In case of change of the maturities published, the CNO will adapt its calculation process to the new data set without any conceptual change.

Euribor Futures contracts

For the short end of the curve, 3-Month Euribor Futures contracts quoted on the Liffe and Eurex are a valuable alternative data source to the IRS swap curve. In order to calculate as of Futures rate present values with maturity of one year[1], two years, three years etc. it is preferable to adjust rates. This adjustment method of the Euribor Futures quotes is commonly used as Euribor Futures market is highly liquid and Futures prices are relatively more reliable than short-term swaps prices.

For the short end of the curve the chosen method[2] consists in calculating the present values of a €1 cash inflow in one year, two years, and three years or even more until a given maturity. The last Futures contract maturity would be determined according to parameters such as market liquidity of the Euribor contracts at each maturity.

Beyond a certain maturity it is adequate to switch to IRS swap quotes when swap market shows better liquidity than Futures market. In current market practice (2013) it is commonly admitted to use Futures rate up to two or three year maturity before switching to swaps rate. The CNO will regularly review the switching maturity, on the basis of Futures market liquidity. This liquidity is assessed by monitoring open positions and transaction volumes on the Futures market[3].

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[1] rolling year

[2] This method is based on Liffe 3Month Euribor contract strip

[3] See Appendix 2: CNO expert decisions for the zero coupon curve calculation, December 2013, chapter “Assessment of the future market liquidity”

As of October 2014,the 3-Month Euribor Futures contracts will be used to calculate the first two Swap Rates Sw_1 and Sw_2.

2. Interpolation method for missing yearly maturities

The selected process is a cubic interpolation[4] in a variable step environment. It allows the continuity of swap rates and of their first two derivatives over time to be preserved for each maturity.

Derivative calculation

We consider swap rates: $\mathbf{Sw_i}$ and have to estimate the derivatives: $\mathbf{P_i = dSw(u)/du}$ **for** $\mathbf{u_i}$.

Δ is time of maturity interval : ($\Delta_{i-1} = u_i - u_{i-1}$, $\Delta_i = u_{i+1} - u_i$),

δ is the swap variation on the interval : ($\delta_{i-1} = Sw_i - Sw_{i-1}$, $\delta_i = Sw_{i+1} - Sw_i$)

***A** is the rate of increase on the interval: ($A_{i-1} = \delta_{i-1} / \Delta_{i-1}$, $A_i = \delta_i / \Delta_i$).*

- For $\mathbf{i = 2}$ to $\mathbf{n-1}$, the derivative $\mathbf{P_i}$ is estimated by the derivative in $\mathbf{u_i}$ of the parabolic function which passes by the known values: $\mathbf{Sw_{i-1}}$, $\mathbf{Sw_i}$ and $\mathbf{Sw_{i+1}}$:

$$P_i = (A_i \times \Delta_{i-1} + A_{i-1} \times \Delta_i) / (\Delta_{i-1} + \Delta_i)$$

- For $\mathbf{i = 1}$ we consider $\mathbf{P_1}$ the derivative in $\mathbf{u_1}$ of the parabolic function which passes by the known values: $\mathbf{SW_1}$, $\mathbf{SW_2}$ and $\mathbf{SW_3}$:

$$P_1 = 2 \times A_1 - P_2$$

- For the maturity $\mathbf{u_n}$, in presence of missing information of the asymptotic behavior of the swap rate curve we assume a zero derivative:

$$P_n = 0$$

Interpolation calculation

For $\mathbf{u_i \leq u \leq u_{i+1}}$ we consider $\mathbf{u = u_i + z \, \Delta_i}$, with $\mathbf{0 \leq z \leq 1}$.

Interpolated values are then calculated by using the formula:

$$Sw(u) = Sw_i \times (1 - z)^3 + (3 \times Sw_i + \Delta_i \times P_i) \times (1 - z)^2 \times z$$
$$+ (3 \times Sw_{i+1} - \Delta_i \times P_{i+1}) \times (1 - z) \times z^2 + Sw_{i+1} \times z^3$$

3. Zero coupon calculation

Short end of the curve

Data issued from Futures contracts are related to the forward rate between two contracts' maturities. These rates represent 3-month forward rates that need to be adjusted due to Futures market practices and its specificities.

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[4] Appendix 1: Detailed interpolation method

Re-treatments: convexity and liquidity adjustments

The future price can be transformed in a future rate by using the following formula[5]:

Futures Rate = 100 – Futures price (R1)

The first re-treatment consists in a rate convexity adjustment, which is inherent from the fact that gains or losses on futures markets translate into immediate margin calls payments due to market fluctuations.

This convexity adjustment is positive by nature and is calculated by using the following formula[6]:

Convexity adjustment = $\frac{1}{2} \times d_1 \times d_2 \times \sigma^2 \times 100$ (R2)

With:

d_1 = Start of period of the Future contract expressed in year fraction

d_2 = End of period of the Future contract expressed in year fraction

σ =Volatility of Future contracts[7]

The second retreatment consists in a liquidity adjustment. For the short end of the curve, 3 month Euribor Futures are used; nevertheless the published zero coupon curve is derived from 6 month swap rates. A liquidity premium must be added to the futures rates. The magnitude of this adjustment is constant and set as the difference between the 3Month Euribor swap rate at 3 years and the 6 Month Euribor swap rate at 3 years8:

Liquidity adjustment = $\mathbf{Sw_3}$ V6M - $\mathbf{Sw_3}$ V3M (R3)

With:

$\mathbf{Sw_3}$ V3M = 3-year Swap rate versus 3-Month Euribor

$\mathbf{Sw_3}$ V6M = 3-year Swap rate versus 6-Month Euribor

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[5] Formula used by clearinghouses for liquidation when contract is due.

[6] "Options, Futures, and Other Derivatives", John C. Hull, technical information are available online: http://media.pearsoncmg.com/intl/fr/ema_fr_hull_optionsfutures_8/notes_techniques/Note01.pdf

[7] See Appendix 2: CNO expert decisions for the zero coupon curve calculation, December 2013, chapter "Choice of the volatility for the convexity adjustment"

[8] See Appendix 2: CNO expert decisions for the zero coupon curve calculation, December 2013, chapter "Choice of the maturity for the liquidity adjustment"

By combining both re-treatments we get the final retreated futures rate:

Adjusted rate = 100 – Futures price – convexity adjustment + liquidity adjustment (R4)

Practice

Once the futures prices are translated into interest rates and re-treated the present value of a given contract (at maturity) can be computed from the present value of the preceding contract (at maturity).

We use the following formula to do so:

Va (i+1) = Va (i) / (1+ Adjusted Rate (i) * n(i) /36000)

n (i) expresses the number of days between the expiration date of the contract i and the expiration date of the contract i +1.

The adjusted rate (i) is calculated as shown above (R4) based on the price of the contract i and on the two adjustments.

Va (0) is the present value at the expiration date of the first contract. Va(0) is calculated from Interbank Money Market. As of October 2014, the 3-Month Euribor will be used as the short interbank money market rate[9].

Va(0) = 1 / (1+ (Short Rate + Liquidity Adjustment) x n(0) / 36000)

Interpolation

Once the present values are calculated for each future contract maturity, the present value due for each calendar year (i.e. one year, two years, etc.) is calculated by interpolating zero coupon-rates. These zero coupon rates represent on one hand the values to be calculated and on the other hand they are considered as parameters in the algorithm used for the calculation of the long end of the curve (see paragraph 3.2).

On the short end of the curve, at three months intervals, the most common method is to interpolate the continuous zero-coupon rate over time.

Practically, if an annual maturity (t_a) falls between two consecutive Futures contract on (t_i) and (t_{i+1}) and with current values (v_i) and (v_{i+1}), we then define:

$$z_i = -\ln(v_i)/t_i$$

$$z_{i+1} = -\ln(v_{i+1})/t_{i+1}$$

We then calculate (z_a) the continuous rate at maturity (t_a) as follows:

$$z_a = z_i + (z_{i+1} - z_i) / (t_{i+1} - t_i) \times (t_a - t_i)$$

$$v_a = \exp(-z_a \times t_a)$$

[9] See Appendix 2: CNO expert decisions for the zero coupon curve calculation, December 2013, chapter “Short rate of the interbank money market”

Finally, the continuous rate of the first annual maturities (1 and 2 years) is reverted to a discrete swap rate to initialize the calculation of the long end of the curve:

Sw(i) = (1- Va(i)) / R (i)

With R(i) = R(i-1) + Va(i)

And R(0) = 0

Long End of the curve

Correspondence between the structure of swap rates and the structure of zero-coupon rate is obtained by writing the parity of the fixed and variable legs.

The fixed leg pays Sw(n) n*times: its value is noted Sw(n) * R(n).

The variable leg allows paying interests on principal. Added to the present value of a hypothetical repayment of principal, it equals the nominal value of the principal. It is therefore equal to the difference between the nominal value of the principal and its forward value at maturity of the swap.

With Sw(n) x R(n) =100 * (1-Va(n))

and R(n)= R(n-1) + Va(n)

hence Sw(n) * R (n-1) + Sw(n) * Va (n) + 100 * Va(n)= 100,

then Va(n) = (100 – Sw(n) x R(n-1)) / (100 + Sw(n))

and Z(n) = (Va(n) ^(-1/n)-1) *100 .

The zero-coupon rate, the present value and the annuity at a given maturity based on the annuity at expiration due from the previous year and the swap rate at the considered maturity can therefore be calculated.

We note R (0) = 0.

All zero coupon rates are then calculated using a recurrence method as follows:

Va (1)= 1/(1 + Sw(1) / 100) ,

Z(1)=Sw(1)

R(1) = Va(1)

Va(2) = (100 – Sw(2) x R(1)) /(100 + Sw(2)) ,

Z(2)= (Va(2)^(-1/2)-1) *100

R(2) = R(1) + Va(2)

Va(i) = (100 – Sw(i) x R(i-1)) /(100 + Sw(i)) ,

$Z(i) = (Va(i)^(-1/i)-1) *100$

$R(i) = R(i-1) + Va(i)$

5. Publication of the Results

Calculated zero-coupon rates are published monthly and are based on swap rates issued on the last business day of the calendar month (TARGET calendar). They are expressed as annual rates with 3 decimals for each maturity from 1 to 60 years.

According to the usual transparency requirements, CNO also publishes:

- A detailed description of the process and calculations.
- Unusual events that may be observed in the process (i.e.: replacement of missing data, corrections,..).

6. Use of the Results

The use of the results is free and is under the user's responsibility. The CNO cannot be held responsible for any consequences of the publication of erroneous figures (e.g.: resulting from erroneous swap rate publications or any other accidental cause, force majeure or due to the malice of a third party) or those resulting from inappropriate use of the published results.

CNO however suggests the following recommendations to use the published results:

- The structure of zero-coupon rate is not calculated in real time. It is used for calculations accounting for strategic analysis or technical or educational applications. Data should not be used for market transactions.
- The method used is copyright-free and can be used by any user, under his own responsibility. Reference to FBA/CNO methodology shall however be made when using this method for any outwards usage, including educational.
- The calculated structure is a credit risk free zero-coupon rates structure. For a discounted cash flow process with embedded credit risk, the appropriate credit spread should be added to the discount rate. The CNO does not issue zero-coupon rate for maturities greater than 60 years due to the lack of swap rates beyond 60 years. The CNO strongly discourages any extrapolation from the published rate structure.
- Zero-coupon rates are published for annual maturities. Interpolation is required when the use relates to non-integer maturities between 1 and 60 years. The interpolation method to be used depends on the desired accuracy. In many cases a linear interpolation is sufficient. If necessary, a cubic interpolation similar to that used above for swap rates can be used. An upper order interpolation is considered to be irrelevant.

Appendix 1: Cubic interpolation methodology

The cubic interpolation method consists in using polynomial functions of degree three with constraints consisting in preserving the continuity of first order derivatives.

A knot can be defined as a set of coordinates (x_i,y_i) with i=1, … ,n .

There are (n-1) polynomial functions (S_i, for i=1 to n-1), S_i being valid on an interval $[x_i, x_{i+1}[$

We are looking for the coefficients of the S_i polynomial functions with the following constraints:

$S_i(x)$ is a 3 degree polynomial	i= 1 to n-1	(1)
$S_i(x_i) = y_i$	i = 1 to n-1	(2)
$S_i(x_{i+1}) = y_{i+1}$	i = 1 to n-1	(3)
$S_i'(x_i) = y'_i$	i = 1 to n-1	(4)
$S_i'(x_{i+1}) = y'_{i+1}$	i = 1 to n-1	(5)

Let's define P_i as a polynomial of degree 3 such that:

$$P_i(t) = a_i.(1 - t)^3 + b_i.(1 - t)^2.t + c_i.(1 - t).t^2 + d_i.t^3 \quad \text{with } a_i, b_i, c_i, d_i \in R \text{ and } t \in [0,1] \qquad (6)$$

Let's also define the affine function f_i such that:

$$f_i(x) = (x - x_i) / (x_{i+1} - x_i) \qquad (7)$$

Using the last two functions, one can show that:

$P_i (f_i (x))$ is also a polynomial of degree 3 (8)

$$f_i(x_i) = 0 \qquad (9)$$

$$f_i(x_{i+1}) = 1 \qquad (10)$$

$$f_i'(x) = 1 / (x_{i+1} - x_i), \text{ for } x \in [x_i, x_{i+1}[\qquad (11)$$

$$P_i'(t) = a_i.(-3).(1 - t)^2 + b_i.((-2).(1 - t).t + (1 - t)^2)+ c_i.((-1). t^2 + (1 - t).2t + d_i.(3).t^2 \qquad (12)$$

$$(P_i (f_i(x)))' = f_i'(x). P_i'(f_i(x)) \qquad (13)$$

From conditions (1) to (5), coefficients a_i, b_i, c_i, d_i may be derived.

From equations (6) and (9), we have:

$S_i(x_i) = a_i.$

Now using relation (2),

$S_i(x_i) = yi$ and $a_i = y_i$ (14)

From equations (6) and (10), we have:

$S_i(x_{i+1}) = d_i.$

Now using relation (3),

$S_i(x_{i+1}) = y_{i+1}$ and $d_i = y_{i+1}$ (15)

From (9), (10), and (11), one can derive:

$$S_i^{'}(x_i) = -3/(x_{i+1} - x_i).a_i + 1/(x_{i+1} - x_i).b_i$$

And

$$S_i^{'}(x_{i+1}) = -1/(x_{i+1} - x_i).c_i + 3/(x_{i+1} - x_i).\ d_i$$

From the system consisting in (4), (5), (14), and (15), it is possible to derive the two coefficients missing:

$$b_i = y'_{i.}(x_{i+1} - x_i) + 3.y_i \quad (16)$$

$$c_i = - y'_{i+1}.(x_{i+1} - x_i) + 3.y_{i+1} \quad (17)$$

The last step consists to set $z = (x - x_i) / (x_{i+1} - x_i)$. Therefore for every i from 1 to n-1, there exists a closed formula for the polynomial $S_i(x)$ such that:

$$\mathbf{S_i (z) = y_i.(1 - z)^3 + (3.y_i + (x_{i+1} - x_i).\ y'_i).(1 – z)^2.z + (3.y_{i+1} – (x_{i+1} - x_i).\ y'_{i+1.}).(1 – z).z^2 + y_{i+1}.z^3}$$

Appendix 2: CNO expert decisions for the zero coupon curve calculation, December 2013

Assessment of the Futures market liquidity

The following table illustrates the liquidity of the Futures market on October 2013, according to the retained indicators: Open positions and Volume.

Ticker	Description	Bid	Ask	Open Pos.	Volume
ERZ3 Comdty	dec13	99,74	99,75	546 236	30 352
ERH4 Comdty	mar-14	99,69	99,69	452 523	13 967
ERM4 Comdty	jun-14	99,64	99,64	362 661	20 263
ERU4 Comdty	sep.-14	99,59	99,59	320 221	34 955
ERZ4 Comdty	dec14	99,52	99,53	309 124	23 344
ERH5 Comdty	mar-15	99,45	99,46	275 916	17 647
ERM5 Comdty	jun-15	99,37	99,37	238 370	10 701
ERU5 Comdty	sep.-15	99,26	99,27	229 247	10 979
ERZ5 Comdty	dec15	99,14	99,15	227 564	15 076
ERH6 Comdty	mar-16	99,00	99,00	162 422	10 086
ERM6 Comdty	jun-16	98,84	98,84	134 308	10 260
ERU6 Comdty	sep.-16	98,66	98,67	142 125	9 661
ERZ6 Comdty	Dec16	98,50	98,50	49 057	7 115
ERH7 Comdty	mar-17	98,34	98,35	56 526	1 276
ERM7 Comdty	jun-17	98,19	98,19	26 094	414
ERU7 Comdty	sep.-17	98,04	98,05	25 032	980
ERZ7 Comdty	Dec17	97,91	97,92	12 182	48
ERH8 Comdty	mar-18	97,81	97,83	8 229	25
ERM8 Comdty	jun-18	97,71	97,73	9 478	6
ERU8 Comdty	sep.-18	97,62	97,64	4 019	72
ERZ8 Comdty	Dec18	97,52	97,54	458	60
ERH9 Comdty	mar-19	97,42	97,44	1	2
ERM9 Comdty	jun-19	97,33	97,43	15	-
ERU9 Comdty	sep.-19	97,26	97,27	-	-

Source: Bloomberg, 17 October 2013

On 17 October 2013, the open positions and the traded volumes on Futures contracts show acceptable liquidity for the maturities up to 2 years (i.e. including the December 2015 contract). Therefore, the zero coupon curve calculation will include contracts up to the shortest future with a maturity above 2 years.

This liquidity will be monitored on a regular basis, and reviewed by the CNO on an annual basis.

Short rate of the interbank money market

As described in chapter 3 “Zero coupon calculation”, a short rate stemming from the interbank money market is necessary to compute the present value at the expiration date of the first contract (Va(0)). Ideally, this rate should be of a maturity equivalent to the first future contract used in the calculation, and for more precision, may be an interpolation of two money market rates with maturities surrounding the first Future contract.

However, the impact of this first rate may be low, and given the flat form of the short money market curve in December 2013, an acceptable simplification is chosen in the calculation of the zero coupon curve. The short rate Va(0) will be defined as the 3-Month Euribor.

The adequacy of this short rate assumption will be monitored on a regular basis, and reviewed by the CNO on an annual basis.

Choice of the volatility for the convexity adjustment

To compute the convexity adjustment, a value is necessary for volatility of the Futures contracts. Different sources may be used to obtain this data, such as observed historical volatility or implied volatility, a flat volatility curve or a volatility surface depending on time and strike or delta. The impact of the convexity adjustment is currently very low (0.005% at 2 years), and a simple approach be preferred for its computation.

Therefore, the chosen source to compute the convexity adjustment is the implied volatility of a 50 Delta option with a maturity close to two years on the Future contract, used as a flat volatility curve for each maturity. The Future contract of two year maturity is chosen considering that the convexity adjustment increases with maturity and has therefore the most impact on the last future used in the zero rate calculation.

The adequacy of this assumption will be monitored on a regular basis, and reviewed by the CNO on an annual basis.

Choice of the maturity for the liquidity adjustment

The liquidity adjustment is derived from the spread between 6-month swap rate and 3-month swap rate, or the 3v6 basis swap spread. In October 2013, 3v6 basis swap spreads for 1 to 10 years range between 10 and 20 basis points. It has been decided to use a flat adjustment set as the spread at 3 years (i.e. between the 3-year Swap rate versus 6-Month Euribor and the 3-year Swap rate versus 3-Month Euribor) considering that 3 year swap rates are the rates with the shortest maturities used in the bootstrapping procedure, and are therefore the most suitable for the liquidity adjustment.

The choice of this maturity is linked to the use of Futures contracts, and will therefore be monitored in adequacy with the liquidity of future markets.